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2010 J. Phys. A: Math. Theor. 43 185204

(http://iopscience.iop.org/1751-8121/43/18/185204)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 185204 (17pp) doi:10.1088/1751-8113/43/18/185204

Polynomial algebras and exact solutions of general quantum nonlinear optical models I: two-mode boson systems

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Received 10 December 2009, in final form 24 February 2010
Published 15 April 2010
Online at stacks.iop.org/JPhysA/43/185204

Abstract
We introduce higher order polynomial deformations of A_1 Lie algebra. We construct their unitary representations and the corresponding single-variable differential operator realizations. We then use the results to obtain exact (Bethe ansatz) solutions to a class of two-mode boson systems, including the Bose–Einstein condensate (BEC) models as special cases. Up to an overall factor, the eigenfunctions of the two-mode boson systems are given by polynomials whose roots are solutions of the associated Bethe ansatz equations. The corresponding eigenvalues are expressed in terms of these roots. We also establish the spectral equivalence between the BEC models and certain quasi-exactly solvable Schördinger potentials.

PACS numbers: 02.20.−a, 02.20.Sv, 03.65.Fd, 42.65.Ky

1. Introduction

Polynomial algebras are nonlinear deformations of Lie algebras and have recently found widespread applications in theoretical physics whereby they appear in diverse topics such as quantum mechanics, Yang–Mills-type gauge theories, quantum nonlinear optics, integrable systems and (quasi-)exactly solvable models, to name a few (see e.g. [1–10]).

One of the reasons for their increasing prevalence stems from the realization that traditional linear Lie algebras describe only a very restrictive subset of linear symmetries and that many physical systems do in fact possess nonlinear symmetries, such as those in which the commutations of the symmetry algebra generators return polynomial terms.

Due to their importance, a number of studies have been undertaken to investigate the mathematical properties of these algebras [11]. In particular, differential realizations of certain quadratic and cubic algebras have been explored in [12, 13] and also in [14, 15] in connection with the theory of quasi-exact integrability [16–18].

In this paper, we introduce a novel class of higher order polynomial deformations of the classical A_1 Lie algebra and construct their unitary representations in terms of boson operators and single-variable differential operators. We will then use the differential realizations of these algebras and the functional Bethe ansatz method (see e.g. [19, 20]) to obtain one of the main results of this paper, that is the exact eigenfunctions and energy eigenvalues of the following class of Hamiltonians:

$$H = \sum_{i}^{2} w_i N_i + \sum_{i,j}^{2} w_{ij} N_i N_j + g\left(a_1^{\dagger s} a_2^r + a_1^s a_2^{\dagger r}\right), \qquad r, s \in \mathbf{N}, \tag{1.1}$$

where and throughout a_i ($a_i^\dagger$) are bosonic annihilation (creation) operators with frequencies w_i, $N_i = a_i^\dagger a_i$ are number operators, w_{ij} and g are real coupling constants. Without loss of generality, in the following we will identify w_{12} with w_{21}. Hamiltonians (1.1) appear in the description of various physical systems of interest such as nonlinear optics [8, 9] and Bose–Einstein condensates (BECs) [21–24]. For instance, the non-diagonal terms in (1.1) describe processes of multi-photon scattering and higher order harmonic generation in quantum nonlinear optics. Let us point out that (1.1) is a two-mode version of the more general multi-mode Hamiltonian considered in [25] in which the quasi-exact solvability of the multi-mode system was established by a different procedure and without giving exact solutions (see also [26] for the case of third-order harmonic generation)[3]. Hamiltonians for the special cases of $s = r = 1$ and $s = 2, r = 1$ have also been studied using the algebraic Bethe ansatz (ABA) method [27].

This paper is organized as follows. In section 2, we propose a class of generalized polynomial $su(1, 1)$ algebras and derive their boson realizations. In section 3, we use these deformed $su(1, 1)$ algebras as base elements to generate higher order polynomial algebras via a Jordan–Schwinger-like construction method. We then identify these algebras as the dynamical algebra of the Hamiltonian (1.1) in section 4 and solve for the eigenvalue problem in general via the functional Bethe ansatz method. In section 5, we present explicit results for the Hamiltonian (1.1) when $r, s \leqslant 2$ and $r = s = 3$. In section 6, we establish the spectral correspondence of these specific models with quasi-exactly solvable (QES) Schrödinger potentials. Finally, we summarize our results in section 7 and discuss further avenues of investigation.

2. Polynomial deformations of $su(1,1)$ algebra

Let k be a positive integer, $k = 1, 2, \ldots$. We start off by proposing a class of polynomial algebras of degree $k - 1$ defined by the commutation relations

$$\begin{aligned} &[Q_0, Q_\pm] = \pm Q_\pm, \\ &[Q_+, Q_-] = \phi^{(k)}(Q_0) - \phi^{(k)}(Q_0 - 1), \end{aligned} \tag{2.1}$$

where

$$\phi^{(k)}(Q_0) = -\prod_{i=1}^{k}\left(Q_0 + \frac{i}{k} - \frac{1}{k^2}\right) + \prod_{i=1}^{k}\left(\frac{i-k}{k} - \frac{1}{k^2}\right) \tag{2.2}$$

is a kth-order polynomial in Q_0. The algebra admits the Casimir operator of the following form:

$$C = Q_- Q_+ + \phi^{(k)}(Q_0) = Q_+ Q_- + \phi^{(k)}(Q_0 - 1). \tag{2.3}$$

[3] We became aware of these three references after submitting our work. We thank one referee for pointing them out.

For $k = 1$ and $k = 2$, (2.1) reduces to the oscillator and $su(1, 1)$ algebras, respectively. Thus, the algebra (2.1) can be viewed as polynomial extensions of the linear $su(1, 1)$ and oscillator algebras.

Similar to the $su(1, 1)$ algebra case, unitary representations of (2.1) are infinite dimensional. In this section, we shall concentrate on the following one-mode boson realization of the algebra:

$$Q_+ = \frac{1}{(\sqrt{k})^k}(a^\dagger)^k, \qquad Q_- = \frac{1}{(\sqrt{k})^k}(a)^k, \qquad Q_0 = \frac{1}{k}\left(a^\dagger a + \frac{1}{k}\right). \tag{2.4}$$

In this realization, the Casimir (2.3) takes the particular value,

$$C = \prod_{i=1}^{k}\left(\frac{i-k}{k} - \frac{1}{k^2}\right). \tag{2.5}$$

We now construct the unitary representations corresponding to the realization (2.4) in the Fock space $\mathcal{H}_b$. There are k lowest weight states,

$$|0\rangle, (a^\dagger)|0\rangle, \ldots, (a^\dagger)^{k-1}|0\rangle. \tag{2.6}$$

Writing these lowest weight states as $|q, 0\rangle$ using the Bargmann index q, we have

$$Q_0|q, 0\rangle = q|q, 0\rangle, \qquad Q_-|q, 0\rangle = 0. \tag{2.7}$$

It follows from (2.3) and (2.5) that $\prod_{i=1}^{k}\left(q + \frac{i-k}{k} - \frac{1}{k^2}\right) = 0$, from which we get

$$q = \frac{1}{k^2}, \frac{k+1}{k^2}, \frac{2k+1}{k^2}, \ldots, \frac{(k-1)k+1}{k^2}. \tag{2.8}$$

This means that the boson realization (2.4) corresponds to the infinite-dimensional unitary representation with particular q values (2.8). In other words, the $\mathcal{H}_b$ decomposes into the direct sum $\mathcal{H}_b = \mathcal{H}_b^{\frac{1}{k^2}} \oplus \cdots \oplus \mathcal{H}_b^{\frac{(k-1)k+1}{k^2}}$ of k irreducible components $\mathcal{H}_b^{\frac{1}{k^2}}, \ldots, \mathcal{H}_b^{\frac{(k-1)k+1}{k^2}}$.

Noting $kq - \frac{1}{k} = 0, 1, \ldots, k-1$ for all the allowed q values given in (2.8), we can write the lowest weight states (2.6) as $|q, 0\rangle = (a^\dagger)^{kq-\frac{1}{k}}|0\rangle$. The general Fock states $|q, n\rangle \sim Q_+^n|q, 0\rangle$ in the irreducible representation space $\mathcal{H}_b^q$ are then given by

$$|q, n\rangle = \frac{a^{\dagger k(n+q-\frac{1}{k^2})}}{\sqrt{\left[k\left(n+q-\frac{1}{k^2}\right)\right]!}}|0\rangle. \tag{2.9}$$

It is easy to show that Q_0, $Q_\pm$ and C act on these states as follows:

$$\begin{aligned}
&Q_0|q, n\rangle = (q+n)|q, n\rangle,\\
&Q_+|q, n\rangle = \prod_{i=1}^{k}\left(n + q + \frac{ik-1}{k^2}\right)^{\frac{1}{2}}|q, n+1\rangle,\\
&Q_-|q, n\rangle = \prod_{i=1}^{k}\left(n + q - \frac{(i-1)k+1}{k^2}\right)^{\frac{1}{2}}|q, n-1\rangle,\\
&C|q, n\rangle = \prod_{i=1}^{k}\left(\frac{i-k}{k} - \frac{1}{k^2}\right)|q, n\rangle,\\
&n = 0, 1, \ldots, \qquad q = \frac{1}{k^2}, \frac{k+1}{k^2}, \ldots, \frac{(k-1)k+1}{k^2}.
\end{aligned} \tag{2.10}$$

3. Polynomial algebras via a Jordan–Schwinger-type construction

The unitary representations of the polynomial algebras discussed in the preceding section are all infinite dimensional. In this section, we shall employ a Jordan–Schwinger-like construction [12, 13], to derive polynomial algebras that have finite-dimensional unitary representations. Towards this end, we consider two mutually commuting polynomial algebras introduced in the preceding section, $\left\{Q_+^{(1)}, Q_-^{(1)}, Q_0^{(1)}\right\}$ of degree $(k_1 - 1)$ and $\left\{Q_+^{(2)}, Q_-^{(2)}, Q_0^{(2)}\right\}$ of degree $(k_2 - 1)$, where $k_1, k_2 = 1, 2, \ldots,$ and introduce new generators,

$$\mathcal{Q}_+ = Q_+^{(1)} Q_-^{(2)}, \qquad \mathcal{Q}_- = Q_+^{(2)} Q_-^{(1)}, \qquad \mathcal{Q}_0 = \tfrac{1}{2}\left(Q_0^{(1)} - Q_0^{(2)}\right). \tag{3.1}$$

We can easily show that $\mathcal{Q}_{0,\pm}$ form a polynomial algebra of degree $(k_1 + k_2 - 1)$ which close under the following commutation relations:

$$\begin{aligned} &[\mathcal{Q}_0, \mathcal{Q}_\pm] = \pm \mathcal{Q}_\pm, \\ &[\mathcal{Q}_+, \mathcal{Q}_-] = \varphi^{(k_1+k_2)}(\mathcal{Q}_0, \mathcal{L}) - \varphi^{(k_1+k_2)}(\mathcal{Q}_0 - 1, \mathcal{L}), \end{aligned} \tag{3.2}$$

where for our purpose we have used (2.5) for the Casimir of $\left\{Q_{0,\pm}^{(i)}\right\}(i = 1, 2)$.

$$\mathcal{L} = \tfrac{1}{2}\left(Q_0^{(1)} + Q_0^{(2)}\right) \tag{3.3}$$

is the central element of the algebra,

$$[\mathcal{L}, \mathcal{Q}_{\pm,0}] = 0, \tag{3.4}$$

and

$$\varphi^{(k_1+k_2)}(\mathcal{Q}_0, \mathcal{L}) = -\prod_{i=1}^{k_1}\left(\mathcal{L} + \mathcal{Q}_0 + \frac{i}{k_1} - \frac{1}{k_1^2}\right)\prod_{j=1}^{k_2}\left(\mathcal{L} - (\mathcal{Q}_0 + 1) + \frac{j}{k_2} - \frac{1}{k_2^2}\right) \tag{3.5}$$

is a $(k_1 + k_2)$th-order polynomial in Q_0 and the central elements $\mathcal{L}$. The Casimir operator of (3.2) is given by

$$\mathcal{C} = \mathcal{Q}_-\mathcal{Q}_+ + \varphi^{(k_1+k_2)}(\mathcal{Q}_0, \mathcal{L}) = \mathcal{Q}_+\mathcal{Q}_- + \varphi^{(k_1+k_2)}(\mathcal{Q}_0 - 1, \mathcal{L}). \tag{3.6}$$

For $k_1 + k_2 = 2$, i.e. $k_1 = k_2 = 1$, the polynomial algebra (3.2) reduces to the linear $su(2)$ algebra. So the algebras defined by (3.2) are polynomial deformations of $su(2)$.

In terms of two sets of mutually commuting boson operators acting on the tensor product of the Fock spaces, we have the realization ($i = 1, 2$ below)

$$Q_+^{(i)} = \frac{1}{(\sqrt{k_i})^{k_i}}(a_2^\dagger)^{k_i}, \qquad Q_-^{(i)} = \frac{1}{(\sqrt{k_i})^{k_i}}(a_2)^{k_i}, \qquad Q_0^{(i)} = \frac{1}{k_i}\left(N_i + \frac{1}{k_i}\right). \tag{3.7}$$

This realization gives rise to finite-dimensional representations of the polynomial algebra (3.2). To show this, let $|q_1, n_1\rangle, |q_2, n_2\rangle$ be the one-mode Fock states of the algebras $\left\{Q_{0,\pm}^{(1)}\right\}, \left\{Q_{0,\pm}^{(2)}\right\}$, respectively, where $n_1, n_2 = 0, 1, \ldots,$ and $q_1 = \frac{1}{k_1^2}, \frac{k_1+1}{k_1^2}, \ldots, \frac{(k_1-1)k_1+1}{k_1^2}$ and $q_2 = \frac{1}{k_2^2}, \frac{k_2+1}{k_2^2}, \ldots, \frac{(k_2-1)k_2+1}{k_2^2}$. The representations of $\{\mathcal{Q}_{0,\pm}\}$ corresponding to the realization (3.7) are then given by the two-mode Fock states $|q_1, n_1\rangle|q_2, n_2\rangle$. Since $\mathcal{L}$ is a central element of the algebra, it must be a constant, denoted as l below, on any irreducible representations. This imposes a constraint on the values of n_1 and n_2,

$$\mathcal{L}|q_1, n_1\rangle|q_2, n_2\rangle = \tfrac{1}{2}(q_1 + n_1 + q_2 + n_2)|q_1, n_1\rangle|q_2, n_2\rangle = l|q_1, n_1\rangle|q_2, n_2\rangle. \tag{3.8}$$

That is, $n_1 + n_2 = 2l - (q_1 + q_2)$. Thus, obviously $2l - q_1 - q_2$ take only positive integer values, i.e.

$$2l - q_1 - q_2 = 0, 1, \ldots. \tag{3.9}$$

It follows that the Fock states corresponding to the realization (3.7) are

$$\begin{aligned}|q_1, q_2, n, l\rangle &= |q_1, n\rangle |q_2, 2l - q_1 - q_2 - n\rangle \\ &= \frac{(a_1^\dagger)^{k_1(n+q_1-\frac{1}{k_1^2})}(a_2^\dagger)^{k_2(2l-q_1-n-\frac{1}{k_2^2})}}{\sqrt{\left(k_1\left(n+q_1-\frac{1}{k_1^2}\right)\right)!}\sqrt{\left(k_2\left(2l-q_1-\frac{1}{k_2^2}-n\right)\right)!}}|0\rangle, \\ n &= 0, 1, \ldots, 2l - q_1 - q_2,\end{aligned} \tag{3.10}$$

noting that $2l - q_1 - q_2$ is always less than or equal to $2l - q_1 - \frac{1}{k_2^2}$. This gives us the $(2l - q_1 - q_2 + 1)$-dimensional irreducible representation of (3.2),

$$\begin{aligned}\mathcal{Q}_0|q_1, q_2, n, l\rangle &= (q_1 - l + n)|q_1, q_2, n, l\rangle, \\ \mathcal{Q}_+|q_1, q_2, n, l\rangle &= \prod_{i=1}^{k_2}\left(2l - q_1 - n - \frac{k_2(i-1)+1}{k_2^2}\right)^{\frac{1}{2}} \\ &\quad\times \prod_{j=1}^{k_1}\left(n + q_1 + \frac{jk_1 - 1}{k_1^2}\right)^{\frac{1}{2}}|q_1, q_2, n+1, l\rangle, \\ \mathcal{Q}_-|q_1, q_2, n, l\rangle &= \prod_{i=1}^{k_2}\left(2l - q_1 - n + \frac{ik_2 - 1}{k_2^2}\right)^{\frac{1}{2}} \\ &\quad\times \prod_{j=1}^{k_1}\left(n + q_1 - \frac{(j-1)k_1+1}{k_1^2}\right)^{\frac{1}{2}}|q_1, q_2, n-1, l\rangle.\end{aligned} \tag{3.11}$$

By using the Fock–Bargmann correspondence,

$$a_i^\dagger \longrightarrow z_i, \qquad a_i \longrightarrow \frac{\mathrm{d}}{\mathrm{d}z_i}, \qquad |n_i\rangle \longrightarrow \frac{z_i^{n_i}}{\sqrt{n_i!}}, \tag{3.12}$$

we can make the following association:

$$|q_1, q_2, n, l\rangle \longrightarrow \frac{z_1^{k_1(n+q_1-\frac{1}{k_1^2})}z_2^{k_2(2l-q_1-n-\frac{1}{k_2^2})}}{\sqrt{\left(k_1\left(n+q_1-\frac{1}{k_1^2}\right)\right)!}\sqrt{\left(k_2\left(2l-q_1-\frac{1}{k_2^2}-n\right)\right)!}}. \tag{3.13}$$

Now since l, q_1, q_2, k_1, k_2 are constants, we can map the states $|q_1, q_2, n, l\rangle$ above to the monomials in $z = z_1^{k_1}/z_2^{k_2}$,

$$\begin{aligned}\Psi_{q_1,q_2,n,l}(z) &= \frac{z^n}{\sqrt{\left(k_1\left(n+q_1-\frac{1}{k_1^2}\right)\right)!}\sqrt{\left(k_2\left(2l-q_1-\frac{1}{k_2^2}-n\right)\right)!}}, \\ n &= 0, 1, \ldots, 2l - q_1 - q_2.\end{aligned} \tag{3.14}$$

The corresponding single-variable differential operator realization of (3.2) takes the following form:

$$\mathcal{Q}_0 = z\frac{\mathrm{d}}{\mathrm{d}z} + q_1 - l,$$
$$\mathcal{Q}_+ = z\frac{(\sqrt{k_2})^{k_2}}{(\sqrt{k_1})^{k_1}}\prod_{j=1}^{k_2}\left(2l - q_1 - \frac{(j-1)k_2+1}{k_2^2} - z\frac{\mathrm{d}}{\mathrm{d}z}\right), \tag{3.15}$$
$$\mathcal{Q}_- = z^{-1}\frac{(\sqrt{k_1})^{k_1}}{(\sqrt{k_2})^{k_2}}\prod_{j=1}^{k_1}\left(z\frac{\mathrm{d}}{\mathrm{d}z} + q_1 - \frac{(j-1)k_1+1}{k_1^2}\right).$$

These differential operators form the same $(2l - q_1 - q_2 + 1)$-dimensional representations in the space of polynomials as those realized by (3.7) in the corresponding Fock space. We remark that because $\prod_{j=1}^{k_1}\left(q_1 - \frac{(j-1)k_1+1}{k_1^2}\right) \equiv 0$ for all the allowed q_1 values, there is no z^{-1} term in $\mathcal{Q}_-$ above, and thus the differential operator expressions (3.15) are non-singular.

4. Exact solution of the two-mode boson systems

We now use the differential operator realization (3.15) to exactly solve the two-mode boson Hamiltonian (1.1).

By means of the Jordan–Schwinger-type construction (3.1) and the realization (3.7), identifying k_1 with s and k_2 with r, we may express the Hamiltonian (1.1) in terms of the generators of the polynomial algebra (3.2),

$$H = \sum_{i}^{2} w_i N_i + \sum_{i,j}^{2} w_{ij} N_i N_j + g\sqrt{s^s r^r}(\mathcal{Q}_+ + \mathcal{Q}_-), \tag{4.1}$$

with the number operators having the following expressions in $\mathcal{Q}_0$ and $\mathcal{L}$:

$$N_1 = s(\mathcal{Q}_0 + \mathcal{L}) - \frac{1}{s}, \qquad N_2 = r(\mathcal{L} - \mathcal{Q}_0) - \frac{1}{r}. \tag{4.2}$$

Keep in mind that $\{\mathcal{Q}_{\mp,0}\}$ in (4.1) as realized by (3.7) (and (3.1)) form the $((2l - q_1 - q_2) + 1)$-dimensional representation of the polynomial algebra (3.2). This representation is also realized by the differential operators (3.15) acting on the $((2l - q_1 - q_2) + 1)$-dimensional space of polynomials with basis $\{1, z, z^2, \ldots, z^{2l-q_1-q_2}\}$. We can thus equivalently represent (4.1) (i.e. (1.1)) as the single-variable differential operator of order $\max\{s, r, 2\}$,

$$H = \sum_{i}^{2} w_i N_i + \sum_{i,j}^{2} w_{ij} N_i N_j + gz\prod_{j=1}^{r} r\left(2l - q_1 - \frac{(j-1)r+1}{r^2} - z\frac{\mathrm{d}}{\mathrm{d}z}\right)$$
$$+ gz^{-1}\prod_{j=1}^{s} s\left(z\frac{\mathrm{d}}{\mathrm{d}z} + q_1 - \frac{(j-1)s+1}{s^2}\right) \tag{4.3}$$

with

$$N_1 = s\left(z\frac{\mathrm{d}}{\mathrm{d}z} + q_1\right) - \frac{1}{s}, \qquad N_2 = r\left(2l - q_1 - z\frac{\mathrm{d}}{\mathrm{d}z}\right) - \frac{1}{r}. \tag{4.4}$$

We will now solve for the Hamiltonian equation

$$H\psi(z) = E\psi(z) \tag{4.5}$$

by using the functional Bethe ansatz method, where $\psi(z)$ is the eigenfunction and E is the corresponding eigenvalue. It is easy to verify

$$Hz^m = z^{m+1} g \prod_{j=1}^{r} r\left(2l - q_1 - \frac{(j-1)r+1}{r^2} - m\right) + \text{lower order terms}, \quad m \in \mathbf{Z}_+. \tag{4.6}$$

This means that the differential operator (4.3) is not exactly solvable. However, it is quasi exactly solvable since it has an invariant polynomial subspace of degree $(2l - q_1 - q_2) + 1$:

$$H\mathcal{V} \subseteq \mathcal{V}, \qquad \mathcal{V} = \text{span}\{1, z, \ldots, z^{2l-q_1-q_2}\}, \qquad \dim\mathcal{V} = 2l - q_1 - q_2 + 1. \tag{4.7}$$

This is easily seen from the fact that when $m = 2l - q_1 - q_2$, the first term on the rhs of (4.6) becomes $z^{2l-q_1-q_2+1} g \prod_{j=1}^{r} r\left(q_2 - \frac{(j-1)r+1}{r^2}\right)$ which vanishes identically for all the allowed q_2 values. We remark that the quasi-exact solvability of the system is connected with its quantum integrability, i.e. with the fact that there exists a quantum operator coinciding with a linear combination of the operators N_1 and N_2 which commutes with the Hamiltonian (1.1).

As (4.3) is a quasi exactly solvable differential operator preserving $\mathcal{V}$, up to an overall factor, its eigenfunctions have the form

$$\psi(z) = \prod_{i=1}^{M}(z - \alpha_i), \tag{4.8}$$

where $M \equiv 2l - q_1 - q_2 (= 0, 1, \ldots)$, and $\{\alpha_i \mid i = 1, 2, \ldots, M\}$ are roots of the polynomial which will be specified later by the associated Bethe ansatz equations (4.14) below. We can rewrite the Hamiltonian (4.3) as

$$H = \sum_{i=1}^{\max\{r,s,2\}} P_i(z)\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^i + P_0(z) \tag{4.9}$$

where

$$\begin{aligned} P_0(z) &= zg\prod_{i=1}^{r} r\left(2l - q_1 - \frac{(i-1)r+1}{r^2}\right) + w_{11}\left(sq_1 - \frac{1}{s}\right)^2 \\ &\quad + w_{22}\left(r(2l-q_1) - \frac{1}{r}\right)^2 + 2w_{12}\left(sq_1 - \frac{1}{s}\right)\left(r(2l-q_1) - \frac{1}{r}\right) \\ &\quad + w_1\left(sq_1 - \frac{1}{s}\right) + w_2\left(r(2l-q_1) - \frac{1}{r}\right) \end{aligned} \tag{4.10}$$

and $P_i(z)$ are the coefficients in front of $\mathrm{d}^i/\mathrm{d}z^i$ in the expansion of (4.3) (see the appendix),

$$\begin{aligned} P_i(z) &= gs^s z^{i-1} \sum_{k=i}^{s}\left(\sum_{l_1<\cdots<l_k}^{s} \prod_{j\neq l_1\neq\cdots\neq l_k}^{s} A_j\right) L_{k,i} + g(-r)^r z^{i+1} \sum_{k=i}^{r}\left(\sum_{l_1<\cdots<l_k}^{r} \prod_{j\neq l_1\neq\cdots\neq l_k}^{r} B_j\right) L_{k,i} \\ &\quad + F\delta_{i,2}z^2 + D\delta_{i,1}z. \end{aligned} \tag{4.11}$$

In the above expression,

$$
\begin{aligned}
A_i &= q_1 - \frac{(i-1)s+1}{s^2},\\
B_i &= -\left(2l - q_1 - \frac{(i-1)r+1}{r^2}\right),\\
L_{k,k} &= 1,\\
L_{k,i} &= \sum_{n_1<\cdots<n_{k-i}}^{k-1} n_1(n_2-1)\cdots(n_{k-i}-(k-i)+1), \qquad i<k,\\
F &= w_{22}r^2 + w_{11}s^2 - 2w_{12}sr,\\
D &= w_{22}r^2\left(1-2\left(2l-q_1-\frac{1}{r^2}\right)\right) + w_{11}s^2\left(1+2\left(q_1-\frac{1}{s^2}\right)\right)\\
&\quad + 2w_{12}rs\left(2(l-q_1)+\frac{1}{s^2}-\frac{1}{r^2}-1\right) + w_1 s - w_2 r.
\end{aligned} \tag{4.12}
$$

Dividing the Hamiltonian equation $H\psi = E\psi$ by ψ gives us

$$
E = \frac{H\psi}{\psi} = \sum_{i=1}^{\max\{r,s,2\}} P_i(z) i! \sum_{l_1<l_2<\cdots<l_i}^{M} \frac{1}{(z-\alpha_{l_1})\cdots(z-\alpha_{l_i})} + P_0(z). \tag{4.13}
$$

The lhs of (4.13) is a constant, while the rhs is a meromorphic function in z with at most simple poles. For them to be equal, we need to eliminate all singularities on the rhs of (4.13). We may achieve this by demanding that the residues of the simple poles, $z = \alpha_i$, $i = 1, 2, \ldots, M$, should all vanish. This leads to the Bethe ansatz equations for the roots $\{\alpha_i\}$:

$$
\sum_{i=2}^{\max\{r,s,2\}} \sum_{l_1<l_2<\cdots<l_{i-1}\neq p}^{M} \frac{P_i(\alpha_p) i!}{(\alpha_p-\alpha_{l_1})\cdots(\alpha_p-\alpha_{l_{i-1}})} + P_1(\alpha_p) = 0, \qquad p = 1, 2, \ldots, M. \tag{4.14}
$$

The wavefunction $\psi(z)$ (4.8) becomes the eigenfunction of H (4.3) in the space $\mathcal{V}$ provided that the roots $\{\alpha_i\}$ of the polynomial $\psi(z)$ (4.8) are the solutions of (4.14).

Some remarks are in order. It is easily seen (from (4.8) and (4.15) below) that $H\psi/\psi$ is regular at $z = \pm\infty$. When (4.14) is satisfied, the rhs of (4.13) is analytic everywhere in the whole complex plane and thus must be a constant by the Liouville theorem. Therefore, the Bethe ansatz equation (4.14) is not only necessary but also a sufficient condition for the rhs of (4.13) to be independent of z.

To obtain the corresponding eigenvalue E, we consider the leading order expansion of $\psi(z)$,

$$
\psi(z) = z^M - z^{M-1}\sum_{i=1}^{M}\alpha_i + \cdots.
$$

It is easy to show that $\mathcal{Q}_{\pm,0}\psi(z)$ have the expansions

$$
\begin{aligned}
\mathcal{Q}_+\psi &= -z^M \frac{(\sqrt{r})^r}{(\sqrt{s})^s}\left[\prod_{j=1}^{r}\left(q_2+1-\frac{(j-1)r+1}{r^2}\right)\right]\sum_{i=1}^{M}\alpha_i + \cdots,\\
\mathcal{Q}_-\psi &= z^{M-1}\frac{(\sqrt{s})^s}{(\sqrt{r})^r}\prod_{j=1}^{s}\left(2l-q_2-\frac{(j-1)s+1}{s^2}\right) + \cdots,\\
\mathcal{Q}_0\psi &= z^M(l-q_2) + \cdots.
\end{aligned} \tag{4.15}
$$

Substituting these expressions into the Hamiltonian equation (4.5) and equating the z^M terms, we arrive at

$$E = w_{11}\left(s(2l-q_2)-\frac{1}{s}\right)^2 + w_{22}\left(rq_2-\frac{1}{r}\right)^2 + 2w_{12}\left(s(2l-q_2)-\frac{1}{s}\right)\left(rq_2-\frac{1}{r}\right)$$
$$+ w_1\left(s(2l-q_2)-\frac{1}{s}\right) + w_2\left(rq_2-\frac{1}{r}\right)$$
$$- g\left[\prod_{j=1}^{r} r\left(q_2+1-\frac{(j-1)r+1}{r^2}\right)\right]\sum_{i=1}^{M}\alpha_i, \qquad (4.16)$$

where $\{\alpha_i\}$ satisfy the Bethe ansatz equations (4.14). This gives the eigenvalue of the two-mode boson Hamiltonian (1.1) with the corresponding eigenfunction $\psi(z)$ (4.8).

5. Explicit examples corresponding to BECs

We will now work out in complete detail the Bethe ansatz equations and energy eigenvalues of the Hamiltonian (1.1) for the special cases of $s, r \leqslant 2$ and $r = s = 3$. These models arise in the description of Josephson tunnelling effects and atom–molecule conversion processes in the context of BECs.

5.1. $s = 1, r = 1$

The Hamiltonian is

$$H = \sum_{i}^{2} w_i N_i + \sum_{i,j}^{2} w_{ij} N_i N_j + g(a_1^\dagger a_2 + a_1 a_2^\dagger). \qquad (5.1)$$

This is the so-called two-coupled BEC model and has been solved in [27] via a different method, i.e. the ABA method. From the general results in the preceding section, in this case, we have $q_1 = q_2 = 1$, which means that $2l - q_1 - q_2 = 2(l-1) = 0, 1, \ldots$. That is, $l - 1 = 0, \frac{1}{2}, 1, \ldots$. The differential operator representation of the Hamiltonian (5.1) is

$$H = P_2(z)\frac{d^2}{dz^2} + P_1(z)\frac{d}{dz} + P_0(z), \qquad (5.2)$$

where

$$P_2(z) = A_{11}z^2,$$
$$P_1(z) = -gz^2 + B_{11}z + g, \qquad (5.3)$$
$$P_0(z) = 2(l-1)gz + D_{11}$$

with

$$A_{11} = w_{11} + w_{22} - 2w_{12} \neq 0,$$
$$B_{11} = w_1 - w_2 + w_{11} + (5-4l)\, w_{22} + (4l-6)\, w_{12}, \qquad (5.4)$$
$$D_{11} = 2(l-1)w_2 + 4(l-1)^2 w_{22}.$$

The Bethe ansatz equations are given by

$$\sum_{i\neq p}^{2(l-1)} \frac{2}{\alpha_i - \alpha_p} = \frac{g + B_{11}\alpha_p - g\alpha_p^2}{A_{11}\alpha_p^2}, \qquad p = 1, 2, \ldots, 2(l-1), \qquad (5.5)$$

and the energy eigenvalues are

$$E = 4w_{11}(l-1)^2 + 2w_1(l-1) - g\sum_{i=1}^{2(l-1)} \alpha_i. \tag{5.6}$$

5.2. $s = 2, r = 1$

The Hamiltonian is

$$H = \sum_i^2 w_i N_i + \sum_{i,j}^2 w_{ij} N_i N_j + g\left(a_1^{\dagger 2} a_2 + a_1^2 a_2^\dagger\right). \tag{5.7}$$

This is the homo-atomic-molecular BEC model and has been solved by the ABA method [27]. Specializing the general results in the preceding section to this case, we have $q_1 = \frac{1}{4}$ or $\frac{3}{4}$, and $q_2 = 1$. The differential operator representation of the Hamiltonian (5.7) is thus

$$H = P_2(z)\frac{d^2}{dz^2} + P_1(z)\frac{d}{dz} + P_0(z) \tag{5.8}$$

where

$$\begin{aligned} P_2(z) &= A_{21}z^2 + 4gz, \\ P_1(z) &= -gz^2 + B_{21}z + 8gq_1, \\ P_0(z) &= g(2l - q_1 - 1)z + D_{21} \end{aligned} \tag{5.9}$$

with

$$\begin{aligned} A_{21} &= 4w_{11} + w_{22} - 4w_{12}, \\ B_{21} &= 2w_1 - w_2 + 2w_{11}(1+4q_1) + w_{22}(3 + 2q_1 - 4l) + w_{12}(-7 - 8q_1 + 8l), \\ D_{21} &= 2w_1\left(q_1 - \tfrac{1}{4}\right) + w_2(2l - q_1 - 1) + 4w_{11}\left(q_1 - \tfrac{1}{4}\right)^2 \\ &\quad + w_{22}(2l - 1 - q_1)^2 + 4w_{12}\left(q_1 - \tfrac{1}{4}\right)(2l - 1 - q_1). \end{aligned} \tag{5.10}$$

The Bethe ansatz equations are

$$\sum_{i\neq p}^{2l-1-q_1} \frac{2}{\alpha_i - \alpha_p} = \frac{8gq_1 + B_{21}\alpha_p - g\alpha_p^2}{\alpha_p(A_{21}\alpha_p + 4g)}, \qquad p = 1, 2, \ldots, 2l - 1 - q_1, \tag{5.11}$$

and the energy eigenvalues are given by

$$E = 2w_1\left(2l - \frac{5}{4}\right) + 4w_{11}\left(2l - \frac{5}{4}\right)^2 - g\sum_{i=1}^{2l-1-q_1} \alpha_i. \tag{5.12}$$

5.3. $s = 2, r = 2$

The Hamiltonian is

$$H = \sum_i^2 w_i N_i + \sum_{i,j}^2 w_{ij} N_i N_j + g\left(a_1^{\dagger 2} a_2^2 + a_1^2 a_2^{\dagger 2}\right). \tag{5.13}$$

This gives another model of the atom–molecule BECs. To our knowledge, this model has not previously been exactly solved. Applying the general results in the preceding section, we have in this case $q_1 = \frac{1}{4}, \frac{3}{4}$ and $q_2 = \frac{1}{4}, \frac{3}{4}$. The differential operator representation of the Hamiltonian (5.13) is

$$H = P_2(z)\frac{d^2}{dz^2} + P_1(z)\frac{d}{dz} + P_0(z), \tag{5.14}$$

where

$$\begin{aligned} P_2(z) &= 4gz^3 + 4A_{22}z^2 + 4gz, \\ P_1(z) &= B_{22}z^2 + D_{22}z + 8gq_1, \\ P_0(z) &= F_{22}z + G_{22}, \end{aligned} \tag{5.15}$$

with

$$\begin{aligned} A_{22} &= w_{11} + w_{22} - 2w_{12}, \\ B_{22} &= 8g(1 + q_1 - 2l), \\ D_{22} &= 2w_1 - 2w_2 + 2w_{11}(1 + 4q_1) + 2w_{22}(3 - 8l + 4q_1) + 8w_{12}(-1 - 2q_1 + 2l), \\ F_{22} &= 4g\left(2l - q_1 - \tfrac{1}{4}\right)\left(2l - q_1 - \tfrac{3}{4}\right), \\ G_{22} &= 2w_1\left(q_1 - \tfrac{1}{4}\right) + 2w_2\left(2l - q_1 - \tfrac{1}{4}\right) + 4w_{11}\left(q_1 - \tfrac{1}{4}\right)^2 \\ &\quad + 8w_{12}\left(q_1 - \tfrac{1}{4}\right)\left(2l - q_1 - \tfrac{1}{4}\right) + 4w_{22}\left(2l - q_1 - \tfrac{1}{4}\right)^2. \end{aligned} \tag{5.16}$$

Note that $(2l - q_1 - q_2)(2l - q_1 + q_2 - 1) \equiv (2l - q_1 - 1/4)(2l - q_1 - 3/4)$ for $q_2 = 1/4, 3/4$. The Bethe ansatz equations read

$$\sum_{i\neq p}^{2l-q_1-q_2} \frac{2}{\alpha_i - \alpha_p} = \frac{8gq_1 + D_{22}\alpha_p - B_{22}\alpha_p^2}{4\alpha_p\left(g\alpha_p^2 + A_{22}\alpha_p + g\right)}, \qquad p = 1, 2, \ldots, 2l - q_1 - q_2, \tag{5.17}$$

and the energy eigenvalues are

$$\begin{aligned} E &= 4w_{11}\left(2l - q_2 - \frac{1}{4}\right)^2 + 4w_{22}\left(q_2 - \frac{1}{4}\right)^2 + 8w_{12}\left(2l - q_2 - \frac{1}{4}\right)\left(q_2 - \frac{1}{4}\right) \\ &\quad + 2w_1\left(2l - q_2 - \frac{1}{4}\right) + 2w_2\left(q_2 - \frac{1}{4}\right) - 4g\left(q_2 + \frac{1}{4}\right)\left(q_2 + \frac{3}{4}\right)\sum_{i=1}^{2l-q_1-q_2} \alpha_i. \end{aligned} \tag{5.18}$$

5.4. $s = 3, r = 3$

The considered examples with $s, r \leqslant 2$ may in principle be treated using the ABA method based on the Lie algebra $su(2)$ (without any polynomial deformations). We now present an explicit example for which the ABA method is not applicable. The Hamiltonian is

$$H = \sum_i^2 w_i N_i + \sum_{i,j}^2 w_{ij} N_i N_j + g\left(a_1^{\dagger 3} a_2^3 + a_1^3 a_2^{\dagger 3}.\right). \tag{5.19}$$

This is a nonlinear optical model with third-order harmonic generation. Specializing the general results in the preceding section to this case, we have $q_1, q_2 = \frac{1}{9}, \frac{4}{9}$ or $\frac{7}{9}$. The differential operator representation of the Hamiltonian (5.19) is

$$H = P_3(z)\frac{d^3}{dz^3} + P_2(z)\frac{d^2}{dz^2} + P_1(z)\frac{d}{dz} + P_0(z), \tag{5.20}$$

where

$$\begin{aligned} P_3(z) &= 27g(-z^4 + z^2) \\ P_2(z) &= A_{33}z^3 + B_{33}z^2 + D_{33}z, \\ P_1(z) &= F_{33}z^2 + G_{33}z + K_{33} \\ P_0(z) &= R_{33}z + S_{33} \end{aligned} \tag{5.21}$$

with

$$
\begin{aligned}
A_{33} &= 9g(18l - 9q_1 - 13),\\
B_{33} &= 9(w_{11} + w_{22} - 2w_{12}),\\
D_{33} &= 9g(9q_1 + 5),\\
F_{33} &= 9g\left(-36l^2 - \tfrac{76}{9} + 34l + 36lq_1 - 9q_1^2 - 17q_1\right),\\
G_{33} &= 3w_1 - 3w_2 + w_{11}(7 + 18q_1) + 2w_{12}(-9 - 18q_1 + 18l) + w_{22}(7 + 18q_1),\\
K_{33} &= 9g\left(q_1 + 9q_1^2 + \tfrac{4}{9}\right),\\
R_{33} &= 27g\left(2l - q_1 - \tfrac{1}{9}\right)\left(2l - q_1 - \tfrac{4}{9}\right)\left(2l - q_1 - \tfrac{7}{9}\right),\\
S_{33} &= 9w_{11}\left(q_1 - \tfrac{1}{9}\right)^2 + 9w_{22}\left(2l - q_1 - \tfrac{1}{9}\right)^2 + 18w_{12}\left(q_1 - \tfrac{1}{9}\right)\left(2l - q_1 - \tfrac{1}{9}\right)\\
&\quad + 3w_1\left(q_1 - \tfrac{1}{9}\right) + 3w_2\left(2l - q_1 - \tfrac{1}{9}\right).
\end{aligned}
\tag{5.22}
$$

The Bethe ansatz equations read

$$
\sum_{i<j\neq p}^{2l-q_1-q_2} \frac{162g\left(\alpha_p^4 - \alpha_p^2\right)}{(\alpha_i - \alpha_p)(\alpha_j - \alpha_p)} + \sum_{i\neq p}^{2l-q_1-q_2} \frac{2\left(A_{33}\alpha_p^3 + B_{33}\alpha_p^2 + D_{33}\alpha_p\right)}{\alpha_i - \alpha_p} = F_{33}\alpha_p^2 + G_{33}\alpha_p + K_{33},
$$
$$
p = 1, 2, \ldots, 2l - q_1 - q_2, \tag{5.23}
$$

and the energy eigenvalues are

$$
\begin{aligned}
E = 9w_{11}&\left(2l - q_2 - \frac{1}{9}\right)^2 + 9w_{22}\left(q_2 - \frac{1}{9}\right)^2\\
&+ 18w_{12}\left(2l - q_2 - \frac{1}{9}\right)\left(q_2 - \frac{1}{9}\right) + 3w_1\left(2l - q_2 - \frac{1}{9}\right)\\
&+ 3w_2\left(q_2 - \frac{1}{9}\right) - 27g\left(q_2 + \frac{2}{9}\right)\left(q_2 + \frac{5}{9}\right)\left(q_2 + \frac{8}{9}\right)\sum_{i=1}^{2l-q_1-q_2} \alpha_i.
\end{aligned}
\tag{5.24}
$$

6. Spectral equivalence with QES Schrödinger potentials

The Hamiltonians in section 5 correspond to second-order differential operators and can be mapped to Schrödinger equations with QES potentials via a suitable similarity transformation and change of variables [28].

Explicitly, if H is written in the following form:

$$
H = P(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left(Q(z) + \frac{1}{2}P'(z)\right)\frac{\mathrm{d}}{\mathrm{d}z} + R(z), \tag{6.1}
$$

then it can be mapped to a Schrödinger operator,

$$
\tilde{H} = -\mathrm{e}^{W(x)} H \mathrm{e}^{-W(x)} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \tag{6.2}
$$

where the variables x and z are related by (we assume $z = z(x)$ is invertible on a certain interval to give $x = x(z)$) [29],

$$
x = x(z) = \pm \int^z \frac{\mathrm{d}y}{\sqrt{P(y)}}, \tag{6.3}
$$

and $W(x)$ is given as

$$
W(x) = \int^{z(x)} \frac{Q(y)}{2P(y)}\mathrm{d}y. \tag{6.4}
$$

The potential function is given by

$$V(x) = \left\{ -R(z) + \frac{1}{2}Q'(z) - \frac{Q(z)(P'(z) - Q(z))}{4P(z)} \right\}\Bigg|_{z=z(x)}. \tag{6.5}$$

Then the solutions of the second-order ODE $H\psi(z) = E\psi(z)$ with eigenvalue E are mapped to solutions of the Schrödinger equation

$$\tilde{H}\tilde{\psi}(x) = \tilde{E}\tilde{\psi}(x) \tag{6.6}$$

with the eigenvalue $\tilde{E} = -E$ and the corresponding Schrödinger wavefunction

$$\tilde{\psi}(x) = \mathrm{e}^{-W(x)}\psi(z(x)). \tag{6.7}$$

We will not discuss the square integrability of the Schrödinger wavefunction $\tilde{\psi}(x)$, but derive the explicit Schrödinger potentials corresponding to the special models in the preceding section.

6.1. $s = 1, r = 1$

For this case, we have

$$\begin{aligned} P(z) &= A_{11}z^2, \\ Q(z) &= -gz^2 + (B_{11} - A_{11})z + g, \\ R(z) &= 2(l-1)gz + D_{11}. \end{aligned} \tag{6.8}$$

From (6.3), we obtain

$$z(x) = \mathrm{e}^{\sqrt{A_{11}}x}. \tag{6.9}$$

The potential is

$$\begin{aligned} V(x) &= \frac{(gz^2 + (A_{11} - B_{11})z - g)(gz^2 + (3A_{11} - B_{11})z - g)}{4A_{11}z^2} \\ &\quad - g(2l-1)z + \frac{B_{11} - A_{11}}{2} - D_{11} \\ &= \frac{g^2}{2A_{11}}\cosh(2\sqrt{A_{11}}x) + g\left(2 - \frac{B_{11}}{A_{11}}\right)\sinh(\sqrt{A_{11}}x) \\ &\quad - (2l-1)g e^{\sqrt{A_{11}}x} - D_{11} + \frac{(A_{11} - B_{11})^2 - 2g^2}{4A_{11}}. \end{aligned} \tag{6.10}$$

6.2. $s = 2, r = 1$

In this case, we have

$$\begin{aligned} P(z) &= A_{21}z^2 + 4gz, \\ Q(z) &= -gz^2 + (B_{21} - A_{21})z + 2g(4q_1 - 1), \\ R(z) &= g(2l - q_1 - 1)z + D_{21}, \end{aligned} \tag{6.11}$$

and $q_1 = \frac{1}{4}$ or $\frac{3}{4}$. From (6.3), we derive

$$z(x) = \frac{2g}{A_{21}}(\cosh(\sqrt{A_{21}}x) - 1). \tag{6.12}$$

The potential is

$$V(x) = \frac{(gz^2 + (A_{21} - B_{21})z + 2g - 8gq_1)(gz^2 + (3A_{21} - B_{21})z + 6g - 8gq_1)}{4(A_{21}z^2 + 4gz)}$$
$$- g(2l - q_1)z + \frac{B_{21} - A_{21}}{2} - D_{21}$$
$$= \frac{g^2}{A_{21}^2} \tanh^2\left(\frac{\sqrt{A_{21}}}{2}x\right) \sinh^2\left(\frac{\sqrt{A_{21}}}{2}x\right)\left[\frac{4g^2}{A_{21}} \sinh^2\left(\frac{\sqrt{A_{21}}}{2}x\right) + 2A_{21} - B_{21}\right]$$
$$+ \frac{1}{4A_{21}}[(A_{21} - B_{21})(3A_{21} - B_{21}) + 8g^2(1 - 2q_1)] \tanh^2\left(\frac{\sqrt{A_{21}}}{2}x\right) - \frac{4(2l - q_1)g^2}{A_{21}}$$
$$\times \sinh^2\left(\frac{\sqrt{A_{21}}}{2}x\right) + \frac{(3 - 8q_1)A_{21} + (4q_1 - 2)B_{21}}{4\cosh^2\left(\frac{\sqrt{A_{21}}}{2}x\right)} + \frac{B_{21} - A_{21}}{2} - D_{21}. \tag{6.13}$$

Here, we have used $(1 - 4q_1)(3 - 4q_1) = 0$ for the two allowed q_1 values $q_1 = \frac{1}{4}$ or $\frac{3}{4}$.

Let us consider the special case of $A_{21} = 0$. In this case,

$$z(x) = gx^2 \tag{6.14}$$

as can be seen from the $A_{21} \to 0$ limit of (6.12). The Schrödinger potential (6.13) then becomes

$$V(x) = \frac{g^4}{16}x^6 - \frac{g^2}{8}B_{21}x^4 + \frac{B_{21}^2 + 8g^2(1 - 4l)}{16}x^2 + q_1B_{21} - D_{21}. \tag{6.15}$$

This is a non-singular sextic potential.

6.3. $s = 2, r = 2$

For this case, we have

$$\begin{aligned} P(z) &= 4gz^3 + 4A_{22}z^2 + 4gz, \\ Q(z) &= B_{22}z^2 + (D_{22} - 4A_{22})z + 8gq_1 - 2g, \\ R(z) &= F_{22}z + G_{22}, \end{aligned} \tag{6.16}$$

and $q_1 = \frac{1}{4}$ or $\frac{3}{4}$. From (6.3), we obtain

$$z(x) = g^{-\frac{1}{3}}\wp(g^{\frac{1}{3}}x; g_2, g_3) - \frac{A_{22}}{3g}, \tag{6.17}$$

where $\wp(x; g_2, g_3)$ is Weierstrass's elliptic function with invariants g_2 and g_3 given by

$$g_2 = \frac{4}{3}g^{\frac{2}{3}}\left(\frac{A_{22}^2}{g^2} - 3\right), \qquad g_3 = \frac{4}{27}A_{22}\left(9 - \frac{2A_{22}^2}{g^2}\right). \tag{6.18}$$

Hereafter we will denote $\wp(x; g_2, g_3)$ simply as $\wp(x)$. The potential is computed as follows:

$$V(x) = (B_{22}z^2 + (D_{22} - 4A_{22})z + 8gq_1 - 2g)\frac{(B_{22} - 12g)z^2 + (D_{22} - 12A_{22})z + 8gq_1 - 6g}{16(gz^3 + A_{22}z^2 + gz)}$$
$$+ (B_{22} - F_{22})z + \frac{D_{22} - 4A_{22}}{2} - G_{22}$$
$$= \sum_{i=1}^{4} c_i \frac{\left(g^{-\frac{1}{3}}\wp(g^{\frac{1}{3}}x) - \frac{A_{22}}{3g}\right)^i}{4\wp'(g^{\frac{1}{3}}x)^2} + (B_{22} - F_{22})g^{-\frac{1}{3}}\wp(g^{\frac{1}{3}}x)$$
$$+ \frac{A_{22}(B_{22} - F_{22})}{3g} + \frac{D_{22} - 4A_{22}}{2} - G_{22}, \tag{6.19}$$

where

$$\begin{aligned}
c_1 &= 2g(4q_1-3)(D_{22}-4A_{22})+2g(4q_1-1)(D_{22}-12A_{22}),\\
c_2 &= 2g(4q_1-3)B_{22}+2g(4q_1-1)(B_{22}-12g)+(D_{22}-4A_{22})(D_{22}-12A_{22}),\\
c_3 &= B_{22}(D_{22}-12A_{22})+(B_{22}-12g)(D_{22}-4A_{22}),\\
c_4 &= B_{22}(B_{22}-12g).
\end{aligned} \tag{6.20}$$

Here, we have used $4(gz^3+A_{22}z^2+gz) = 4\wp(g^{\frac{1}{3}}x)^3 - g_2\wp(g^{\frac{1}{3}}x) - g_3 = \wp'(g^{\frac{1}{3}}x)^2$ and $(1-4q_1)(3-4q_1) = 0$ for the two allowed q_1 values $q_1 = \frac{1}{4}$ or $\frac{3}{4}$.

7. Discussion

Let us now quickly summarize the work. We began by constructing the boson representation of a class of $su(1,1)$ polynomially deformed algebras (2.1), deriving their infinite-dimensional Fock space realization and lowest weight state parametrization. We then used the Jordan–Schwinger-like construction to get the polynomial algebra (3.2) which possesses finite-dimensional irreducible representations. We used the differential realization of (3.2) to rewrite the Hamiltonian (1.1) as QES differential operators acting on the finite-dimensional monomial space. The exact eigenfunctions and eigenvalues of the Hamiltonian were then found by employing the functional Bethe ansatz technique. As examples, we provided some explicit expressions for the BEC models which correspond to the $r, s \leqslant 2$ cases of (1.1) and established the spectral correspondence of these specific models with QES Schrödinger potentials.

In deriving our results, we showed that in general the Hamiltonians defined in (1.1) are QES differential operators of order 3 or higher. This paper provides an algebraization of such higher order QES differential operators and unravels the dynamical polynomial algebra symmetry of (1.1). It also shows that the functional Bethe ansatz method provides a simple way to find exact eigenvalues and eigenfunctions of such higher order differential operators.

There are a number of extensions that we plan to pursue in this line of investigation. First, we note that the Jordan–Schwinger-like construction can be extended straightforwardly to study other nonlinear quantum optical models such as the general multi-mode boson Hamiltonians of the form

$$H = \sum_{i}^{k+k'} w_i N_i + \sum_{i,j}^{k+k'} w_{ij} N_i N_j + g\left(a_1^{\dagger m_1}\cdots a_k^{\dagger m_k} a_{k+1}^{m_{k+1}}\cdots a_{k+k'}^{m_{k+k'}} + a_1^{m_1}\cdots a_k^{m_k} a_{k+1}^{\dagger m_{k+1}}\cdots a_{k+k'}^{\dagger m_{k+k'}}\right). \tag{7.1}$$

Results on this and other models of physical interest will be presented in future publications.

Acknowledgments

This work was supported by the Australian Research Council. The authors would like to thank Ryu Sasaki for very valuable comments and suggestions which lead to significant improvement of the presentation of the paper.

Appendix

In this appendix, we work out the expansion coefficients in front of $\frac{d^i}{dz^i}$ in the expansion of $\prod_{i=1}^{m}\left(z\frac{d}{dz}+A_i\right)$.

First, we see that

$$\prod_{i=1}^{m}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+A_i\right)=\prod_{i=1}^{m}A_i+\left(\sum_{j_1=1}^{m}\prod_{i\neq j_1}^{m}A_i\right)z\frac{\mathrm{d}}{\mathrm{d}z}+\left(\sum_{j_1<j_2}^{m}\prod_{i\neq j_1\neq j_2}^{m}A_i\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^2$$
$$+\cdots+\left(\sum_{j_1<j_2<\cdots<j_m}^{m}\prod_{i\neq j_1\cdots\neq j_m}^{m}A_i\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^m, \tag{A.1}$$

since

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^k=z^k\frac{\mathrm{d}^k}{\mathrm{d}z^k}+\left(\sum_{n=1}^{k-1}n\right)z^{k-1}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}+\left(\sum_{n_1<n_2}^{k-1}n_1(n_2-1)\right)z^{k-2}\frac{\mathrm{d}^{k-2}}{\mathrm{d}z^{k-2}}+\cdots$$
$$=\sum_{i=1}^{k}L_{k,i}z^i\frac{\mathrm{d}^i}{\mathrm{d}z^i} \tag{A.2}$$

where

$$L_{k,k}=1$$
$$L_{k,i}=\sum_{n_1<\cdots<n_{k-i}}^{k-1}n_1(n_2-1)\cdots(n_{k-i}-(k-i)+1),\qquad i<k. \tag{A.3}$$

We can regroup equation (8.1) as

$$\prod_{i=1}^{m}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+A_i\right)=\prod_{i=1}^{m}A_i+\left(L_{1,1}\left(\sum_{j_1=1}^{m}\prod_{i\neq j_1}^{m}A_i\right)\right.$$
$$\left.+\cdots+L_{m,1}\left(\sum_{j_1<j_2<\cdots<j_m}^{m}\prod_{i\neq j_1\cdots\neq j_m}^{m}A_i\right)\right)z\frac{\mathrm{d}}{\mathrm{d}z}+\left(L_{2,2}\left(\sum_{j_1<j_2}^{m}\prod_{i\neq j_1\neq j_2}^{m}A_i\right)\right.$$
$$\left.+\cdots+L_{m,2}\left(\sum_{j_1<j_2<\cdots<j_m}^{m}\prod_{i\neq j_1\cdots\neq j_m}^{m}A_i\right)\right)z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\text{higher order terms}$$
$$=\prod_{i=1}^{m}A_i+\sum_{i=1}^{m}\sum_{k=i}^{m}\left(\sum_{l_1<\cdots<l_k}^{m}\prod_{j\neq l_1\neq\cdots\neq l_k}^{m}A_j\right)L_{k,i}z^i\frac{\mathrm{d}^i}{\mathrm{d}z^i}. \tag{A.4}$$

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